Fourier Coefficients of Hilbert Modular Forms at Cusps

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March 28, 2022

Modular forms

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be a holomorphic normalised newform of weight k and level N.
- We denote the field generated by all the Fourier coefficients of f by $\mathbb{Q}(f)$

Fourier expansion at cusps

Let $\mathfrak{a}$ be a cusp of $\Gamma_0(N)\backslash\mathbb{H}$. This is equivalent to a rational number

$$\mathfrak{a} = \frac{a}{L}, \text{ where } L|N \text{ and } (a, N) = 1.$$

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$$f|_k\sigma(z) = \sum_{n\geq 0} a_f(n;\sigma)e^{2\pi i n z/w(\mathfrak{a})},$$

where $w(\mathfrak{a}) = N/(L^2, N)$ is the width of the cusp.

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- The q-expansion principle implies that the Fourier coefficients at any cusp lie in the number field $\mathbb{Q}(f)(\zeta_N)$

A question about the Fourier coefficients of $f|_k g$

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A question about the Fourier coefficients of $f|_k g$

- Let f be a normalised newform of level N and weight k and $g \in \mathrm{SL}_2(\mathbb{Z})$; what is the number field that the Fourier coefficients of $f|_k g$ generate?
- Can one write down an explicit subfield of $\mathbb{Q}(f)(\zeta_N)$, depending on the entries of g, which contains all the Fourier coefficients of $f|_k g$?

The answer

Theorem (Brunault & Neururer, 2020)

Let f be a normalised newform on $\Gamma_0(N)$ with weight k. Let $\mathbb{Q}(f)$ be the field generated by all the Fourier coefficients of f. Let $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then the Fourier coefficients of $f|_k\sigma$ lie in the cyclotomic extension $\mathbb{Q}(f)(\zeta_{N'})$ where $N' = N/(cd, N)$.

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- This result holds more generally for modular forms on $\Gamma_1(N)$
- The proof is purely classical

Hilbert modular forms I

Hilbert modular forms I

- Let F be a totally real number field of degree n with narrow class group of size h and ring of integers $\mathcal{O}_F$. Let $\mathfrak{n}$ denote a fixed integral ideal of $\mathcal{O}_F$
- For $\mu = 1, ..., h$, we define the congruence subgroup $\Gamma_\mu(\mathfrak{n})$ of $\mathrm{GL}_2(F)$ as

$$\Gamma_\mu(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in \mathcal{O}_F, b \in (t_\mu)^{-1}\mathfrak{D}_F^{-1}, c \in \mathfrak{n}t_\mu\mathfrak{D}_F, ad{-}bc \in \mathcal{O}_F^\times \right\},$$

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$$f_\mu(z) = \sum_{\xi \in (t_\mu \mathcal{O}_F)_+ \cup \{0\}} a_\mu(\xi) e^{2\pi i \,\mathrm{Tr}(\xi z)}.$$

Hilbert modular forms II

- A Hilbert newform of weight k and level $\mathfrak{n}$ is a tuple $\mathbf{f} = (f_1, ..., f_h)$ where f_μ is a Hilbert cuspform for $\Gamma_\mu(\mathfrak{n})$ and $\mathbf{f}$ does not come from lower level and is a Hecke eigenform

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- We define

$$c_\mu(\xi; f_\mu) = N(t_\mu \mathcal{O}_F)^{-k_0/2} a_\mu(\xi) \xi^{(k_0\mathbf{1}-k)/2},$$

where $k_0 = \max\{k_1, ..., k_n\}$ and $k_0\mathbf{1} = (k_0, ..., k_0)$

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- We let $\mathbb{Q}(\mathbf{f})$ denote the number field generated by $c_\mu(\xi; f_\mu)$ as ξ varies over F and μ varies over $1 \leq \mu \leq h$ (Shimura, 1978).

Main result

- Let $\mathbf{f} = (f_1, ..., f_h)$ be a normalised newform of level $\mathfrak{n}$ and weight $k = (k_1, ..., k_n)$ and $\sigma \in \Gamma_\mu(1)$; what is the explicit cyclotomic extension (depending on σ) of $\mathbb{Q}(\mathbf{f})$ which contains all the Fourier coefficients of $f_\mu||_k\sigma$?
- Let $\mathbb{Q}(\mathbf{f}, \mu, \sigma)$ denote the field generated by $c_\mu(\xi; f_\mu||_k\sigma)$ as ξ varies over F.

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Theorem

Let $\mathbf{f} = (f_1, ..., f_h)$ be a normalised cuspidal Hilbert newform of level $\mathfrak{n}$ and weight $k = (k_1, ..., k_n)$ with $k_1 \equiv ... \equiv k_n \pmod 2$. Let $1 \leq \mu \leq h$ and $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\mu(1)$. Then $\mathbb{Q}(\mathbf{f}, \mu, \sigma)$ lies in the number field $\mathbb{Q}(\mathbf{f})(\zeta_{N_0})$ where N_0 is the integer such that $N_0\mathbb{Z} = \mathfrak{n}/(cdt_\mu^{-1}\mathfrak{D}_F^{-1}, \mathfrak{n}) \cap \mathbb{Z}$.

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$$\tau(W_v(g)) = W_{{}^\tau\Pi_v}(a(\alpha_\tau)g) \text{ (Raghuram \& Tanabe, 2011)}$$

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- So if $\Pi_v \otimes |\ |^{k_0/2} \cong {}^\tau(\Pi_v \otimes |\ |^{k_0/2})$, we have that

$$\tau\big(W_v(g)\big)\tau\big(|\det(g)|_v^{k_0/2}\big) = W_v(a(\alpha_\tau)g)|\det(g)|_v^{k_0/2}.$$

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- We have $f_\mu||_k\sigma$ has a Fourier expansion of the form

$$f_\mu||_k\sigma(z) = \sum_{((t_\mu\mathcal{O}_F)\mathfrak{w}(\sigma,\mathfrak{n})^{-1})_+} a_\mu(\xi;\sigma)e^{2\pi i\,\mathrm{Tr}(\xi z)},$$

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Let $\xi \in F^\times$. Then

$$W_\phi(a(\xi)g_z\iota_{\mathsf{f}}(\sigma^{-1})x_\mu) = \begin{cases} y^{k/2}a_\mu(\xi;\sigma)e(\mathrm{Tr}(\xi z)), \text{ if } \xi \in ((t_\mu\mathcal{O}_F)\mathfrak{w}(\sigma,\mathfrak{n})^{-1})_+ \\ 0, \text{ otherwise.} \end{cases}$$

Overview of main result proof

- From key result II we have

$$\begin{aligned} a_\mu(\xi;\sigma) &= y^{-k/2} e^{-2\pi i \operatorname{Tr}(\xi z)} W_\phi(a(\xi) g_z \iota_{\mathsf{f}}(\sigma^{-1}) x_\mu) \\ &= \xi^{k/2} \prod_{v<\infty} W_v(a(\xi)\iota_{\mathsf{f}}(\sigma^{-1}) x_{\mu,v}), \text{ from evaluating } W_\infty \end{aligned}$$

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- Thus

$$c_\mu(\xi; f_\mu||_k\sigma) = N(t_\mu \mathcal{O}_F)^{-k_0/2} \xi^{k_0 \mathbf{1}/2} \prod_{v<\infty} W_v(a(\xi)\iota_{\mathsf{f}}(\sigma^{-1}) x_{\mu,v})$$

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- Suppose that $\tau\in\mathrm{Aut}(\mathbb{C})$ fixes $\mathbb{Q}(\mathbf{f})(\zeta_{N_0})$ then use Key result I to find $\tau(W_v(a(\xi)\iota_{\mathsf{f}}(\sigma^{-1})x_{\mu,v}))$

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- Show

$$
\frac{\tau(c_\mu(\xi; f_\mu||_k \sigma))}{c_\mu(\xi; f_\mu||_k \sigma)} = 1
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Closing remarks

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- Can one generalise these results to the case of non-trivial central character?
- Can one write an algorithm to compute the Fourier coefficients of Hilbert newforms at cusps?

Thank You

Any Questions?